Nonsmooth domain optimization for elliptic equations with unilateral conditions

A. Khludnev a,*, A. Leontiev b, J. Herskovits c

a Lavrentyev Institute of Hydrodynamics of the Russian Academy of Sciences, Novosibirsk, 630090, Russia
b Instituto de Matemática, Universidade Federal do Rio de Janeiro, 21954 970, Rio de Janeiro-RJ, Brazil
c Programa de Engenharia Mecânica, COPPE/Universidade Federal do Rio de Janeiro, 21954 970, Rio de Janeiro-RJ, Brazil

Received 30 July 2002

Abstract

In the paper we consider elliptic boundary problems in domains having cuts (cracks). The non-penetration condition of inequality type is prescribed at the crack faces. A dependence of the derivative of the energy functional with respect to variations of crack shape is investigated. This shape derivative can be associated with the crack propagation criterion in the elasticity theory. We analyze an optimization problem of finding the crack shape which provides a minimum of the energy functional derivative with respect to a perturbation parameter and prove a solution existence to this problem.

© 2003 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Keywords: Optimization; Nonsmooth domain; Crack; Derivative of energy functional

* Corresponding author.
E-mail addresses: khlud@hydro.nsc.ru (A. Khludnev), anatoli99@ig.com.br (A. Leontiev), jose@com.ufrj.br (J. Herskovits).

0021-7824/03/$ – see front matter © 2003 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.
doi:10.1016/S0021-7824(03)00005-9
1. Introduction

Classical linear approach to the crack problem is characterized by the equality type boundary condition at the crack faces. This approach does not exclude the mutual penetration of crack faces what has been discussed in many works. In contrast with this linear approach, there are nonlinear models widely presented in [6] which do not allow the mutual penetration between crack faces, and consequently, from the standpoint of applications these nonlinear models are more suitable.

The well-known Griffith criterion states that a crack propagation occurs provided that the derivative of the energy functional with respect to the crack length reaches some critical value. This derivative depends, in particular, on the crack shape which in fact means the dependence on the domain shape. In this work we analyze the dependence of the derivative of the energy functional on the domain shape for the nonlinear crack theory. Our goal is to find a domain shape providing a minimal derivative of the energy functional. We prove a solution existence to this problem.

Boundary value problems describing nonlinear cracks with the non-penetration conditions for many constitutive laws can be found in [6]. In this case inequality type restrictions are imposed on the solution which implies the nonlinearity of the analyzed problems. Numerical analysis of similar problems was fulfilled in [12].

Dependence of solutions on parameters for different domain perturbations has been investigated in many works. The case of smooth domains was considered in [18]. Nonsmooth domains are analyzed in [2]. Results on differentiability of the energy functional for domains with cuts (cracks) in elastic problems can be found in [13,17]. General problems related to solution singularities for nonsmooth domains are presented in [4,5,10,14,15]. As for concrete solutions and theory applications we refer the reader to [3,16].

The differentiability of the energy functional for the nonlinear crack theory is analyzed in [7,8]. Optimal control in boundary problems for elastic bodies with inequality type restrictions imposed on the solutions can be found in [9]. As for the classical approaches to optimal control problems in the linear elasticity we refer to [1].

In this section, we consider a perturbation of the equilibrium problem for an elastic membrane with nonlinear cracks through a family of domains and present the corresponding shape derivative of the energy functional. In Section 2 we formulate an optimal control problem to find a domain shape with a needed property. Examples and related optimal control problems are presented in Section 3.

We should note at this point that the idea of the paper can be applied to other linear and nonlinear elliptic problems for domains with cracks.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\Gamma_0$, and $\Gamma_0^c$ be a smooth curve without selfintersections such that $\Gamma_0^c \subset \Omega$. Denote $\Omega_0 = \Omega \setminus \overline{\Gamma_0^c}$. It is assumed that a membrane occupies the domain $\Omega_0$, and $\Gamma_0^c$ corresponds to the cut (crack) in the membrane. The equilibrium problem for the membrane having the crack can be formulated as follows. We have to find a displacement $u$ such that

$$-\Delta u = f \quad \text{in } \Omega_0,$$  \hspace{1cm} (1.1)
199

\begin{align}
  u &= 0 \quad \text{on } \Gamma_0, \\
  [u] &\geq 0, \quad \frac{\partial u}{\partial v} \leq 0, \quad \left[ \frac{\partial u}{\partial v} \right] = 0, \quad [u] \frac{\partial u}{\partial v} = 0 \quad \text{on } \Gamma_0^c. 
\end{align}

The brackets \([v] = v^+ - v^-\) mean the jump of the function \(v\) through \(\Gamma_0^c\), and \(v^\pm\) fit the positive and negative crack faces \(\Gamma_0^{c\pm}\) with respect to the unit normal vector \(v\) on \(\Gamma_0^c\).

The function \(f \in C^1_{\text{loc}}(\mathbb{R}^2)\) is given.

Problem (1.1)–(1.3) is uniquely solvable, and it admits the variational formulation. Namely, let \(H^{1,0}(\Omega_0)\) be the Sobolev space of functions having the first square integrable derivatives and equal to zero at the external boundary \(\Gamma_0\). Consider the closed convex set of admissible displacements:

\[ K_0 = \{ v \in H^{1,0}(\Omega_0) \mid [v] \geq 0 \text{ on } \Gamma_0^c \}. \]

Then the problem (1.1)–(1.3) is equivalent to minimization of the functional

\[ \frac{1}{2} \int_{\Omega_0} |\nabla v|^2 - \int_{\Omega_0} f v \]

over the set \(K_0\), and it can be written in the variational inequality form

\[ u \in K_0; \quad \int_{\Omega_0} \nabla u (\nabla \tilde{u} - \nabla u) \geq \int_{\Omega_0} f (\tilde{u} - u) \quad \forall \tilde{u} \in K_0. \]

We can define the energy functional;

\[ E(\Omega_0) = \frac{1}{2} \int_{\Omega_0} |\nabla u|^2 - \int_{\Omega_0} f u \]

for the problem (1.4).

Consider next the family of perturbations of the domain \(\Omega_0\):

\[ x = \varphi_\varepsilon(y), \quad y \in \overline{\Omega}. \]

We assume that \(\varphi_\varepsilon\) establishes a one-to-one correspondence between \(\overline{\Omega}\) and \(\varphi_\varepsilon(\overline{\Omega})\), \(\varphi_0(y) = y\), and the Jacobian \(|\partial \varphi_\varepsilon(y)/\partial y|\) is positive. Also, the smoothness \(\varphi, \varphi^{-1} \in C^2(-\varepsilon_0, \varepsilon_0; C^1_{\text{loc}}(\mathbb{R}^2))\) is assumed, where \(\varepsilon_0 > 0\) is a given number. For any fixed \(\varepsilon \in (-\varepsilon_0, \varepsilon_0)\) we can consider the perturbation of the problem (1.1)–(1.3). In fact, let \(\Gamma_\varepsilon = \varphi_\varepsilon(\Gamma_0), \Gamma_\varepsilon^c = \varphi_\varepsilon(\Gamma_0^c), \Omega_\varepsilon = \varphi_\varepsilon(\Omega_0)\). Then the perturbed problem can be formulated in the following form. We have to find a displacement \(u_\varepsilon\) such that
\[ \begin{align*}
-\Delta u^\varepsilon &= f \quad \text{in } \Omega^\varepsilon, \\
u^\varepsilon &= 0 \quad \text{on } \Gamma^\varepsilon, \\
\alpha [u^\varepsilon] &\geq 0, \quad \frac{\partial u^\varepsilon}{\partial \nu^\varepsilon} \leq 0, \quad \left[ \frac{\partial u^\varepsilon}{\partial \nu^\varepsilon} \right] = 0, \quad \left[ u^\varepsilon \right] \frac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = 0 \quad \text{on } \Gamma^\varepsilon. 
\end{align*} \] (1.5)

Here \( \nu^\varepsilon \) is the unit normal vector to \( \Gamma^\varepsilon \). As before, the problem (1.5)–(1.7) admits the variational formulation. If

\[ K^\varepsilon = \{ v \in H^{1,0}(\Omega^\varepsilon) \mid [v] \geq 0 \text{ on } \Gamma^\varepsilon \} \]

is the set of admissible displacements then the relations (1.5)–(1.7) are equivalent to the variational inequality

\[ u^\varepsilon \in K^\varepsilon: \quad \int_{\Omega^\varepsilon} \nabla u^\varepsilon (\nabla \tilde{u} - \nabla u^\varepsilon) \geq \int_{\Omega^\varepsilon} f (\tilde{u} - u^\varepsilon) \quad \forall \tilde{u} \in K^\varepsilon. \] (1.8)

The Sobolev space \( H^{1,0}(\Omega^\varepsilon) \) is introduced similar to \( H^{1,0}(\Omega_0) \), in particular, functions from \( H^{1,0}(\Omega^\varepsilon) \) are equal to zero on \( \Gamma^\varepsilon \).

Observe that the problem (1.5)–(1.7) (or (1.8) what is the same) is the problem (1.1)–(1.3) as \( \varepsilon = 0 \).

For the future considerations, it is necessary to introduce the vector-field \( V(y) \) by the formula:

\[ V(y) = \left. \frac{d\phi^\varepsilon(y)}{d\varepsilon} \right|_{\varepsilon=0}. \]

This vector-field \( V(y) = (V^1(y), V^2(y)) \) is defined, in particular, in the domain \( \Omega \).

As it was proved in [7] in more general setting the energy functional of the problem (1.8), i.e., the functional

\[ E(\Omega^\varepsilon) = \frac{1}{2} \int_{\Omega^\varepsilon} |\nabla u^\varepsilon|^2 - \int_{\Omega^\varepsilon} fu^\varepsilon \]

has the derivative \( E' = dE(\Omega^\varepsilon)/d\varepsilon|_{\varepsilon=0} \) with respect to \( \varepsilon \) as \( \varepsilon = 0 \). Moreover, the following formula holds:

\[ E' = \int_{\Omega_0} \left\{ \frac{1}{2} |\nabla u|^2 \text{ div } V - u \cdot u \cdot p V P \right\} - \int_{\Omega_0} u \text{ div } (fV). \] (1.9)

Note that if the perturbation \( \phi^\varepsilon \) describes the crack length change, the formula (1.9) provides the derivative of the energy functional with respect to the crack length. Such a derivative is used in the classical Griffith criterion to answer the question on the crack propagation for elastic bodies.
All details concerning the formula (1.9) can be found in [7] and therefore we do not provide its derivation. Similar formula for a case of the linear boundary conditions at the crack faces

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_0^{c \pm}$$

can be found in [11].

2. Optimal domain shapes

In the sequel we assume that the curve $\Gamma_0^c$ coincides with the graph of the function $y_2 = \psi(y_1)$, $y_1 \in (a,b)$. The function $\psi$ will be a control function and it will be chosen from a suitable functional space. For any fixed $\psi$ we can find the derivative (1.9) and obtain therefore that $E' = E'(\psi)$.

Introduce the Sobolev space $H^2(a,b)$ with the elements having derivatives up to the second order which are square integrable in $(a,b)$. Let $\Psi \subset H^2(a,b)$ be a bounded and weakly closed set. We assume that for any $\psi \in \Psi$ the graph of the function $y_2 = \psi(y_1)$, $y_1 \in (a,b)$, belongs to the domain $\Omega$.

Consider the optimal control problem:

$$\min_{\psi \in \Psi} E'(\psi). \quad (2.10)$$

This means that we want to find the crack shape which guarantees the minimal value of the derivative $E'(\psi)$ on the set $\psi \in \Psi$. Hence we, in fact, want to obtain the optimal domain shape. The solution of the problem (2.10) gives the most dangerous crack shape provided that $\psi_\varepsilon$ describes the crack length change, and the classical Griffith criterion is used for the crack propagation. Indeed, it is easy to show that in this case, i.e., in the situation when $\psi_\varepsilon$ describes the crack length change, $E'(\psi) \leq 0$ for all $\psi \in \Psi$. According to the Griffith criterion there exists a constant $\kappa < 0$ such that the equality $E'(\psi) = \kappa$ implies a crack propagation. On the other hand, the crack shape $y_2 = \psi_0(y_1)$ with the inequality $E'(\psi) > \kappa$ provides a stable crack, i.e., there is no propagation in this case. Let $\psi_0$ be a solution of the problem (2.10). In this case we can say that the crack shape $y_2 = \psi_0(y_1)$ is not dangerous if $\psi_0$ satisfies the inequality $E'(\psi_0) > \kappa$.

The aim of the arguments below is to provide some properties of solutions to problems like (1.1)–(1.3) in order to prove a solution existence of the problem (2.10). We first establish an auxiliary result concerning the strong convergence of solutions which guarantees the continuity of the derivative with respect to the crack shape and consequently with respect to the domain shape.

Assume that we consider the family of cracks described by the graphs $\Gamma_0^c$ of functions $y_2 = \delta \psi(y_1)$, $y_1 \in (a,b)$, where $\delta$ is a small parameter converging to zero and $\psi \in \Psi$ is a fixed element. We want to prove that solutions of problems like (1.1)–(1.3) corresponding to the parameter $\delta$ converge strongly as $\delta \to 0$. 
Let \( \Omega_\delta^0 \) be a domain corresponding to \( \Gamma_{c\delta}^c \), i.e., \( \Omega_\delta^0 = \Omega \setminus \overline{\Gamma_{c\delta}^c} \). In this case for \( \delta = 0 \) we have \( \Omega_0^0 = \Omega_0 \), \( \Gamma_0^c = (a, b) \times \{0\} \). So, in fact, we consider the perturbation of the crack shape through the parameter \( \delta \).

Denote by:

\[
K_{\delta} = \{ v \in H^1(\Omega_\delta^0) \mid [v] \geq 0 \text{ on } \Gamma_\delta^c \}
\]

the set of admissible displacements and consider a solution \( u_\delta \) of the problem

\[
\int_{\Omega_\delta^0} \nabla u_\delta \left( \nabla \bar{u}^\delta - \nabla u^\delta \right) \geq \int_{\Omega_\delta^0} f \left( \bar{u}^\delta - u^\delta \right) \quad \forall \bar{u}^\delta \in K_{\delta}. \tag{2.11}
\]

Analogously, for \( \delta = 0 \) we can consider the solution \( u \) of the unperturbed problem

\[
\int_{\Omega_0^0} \nabla u \left( \nabla \bar{u} - \nabla u \right) \geq \int_{\Omega_0^0} f \left( \bar{u} - u \right) \quad \forall \bar{u} \in K_0 \tag{2.12}
\]

with a convex and closed set of admissible displacements

\[
K_0 = \{ v \in H^{1,0}(\Omega_0^0) \mid [v] \geq 0 \text{ on } \Gamma_0^c \}.
\]

It is possible to establish a one-to-one correspondence between the domains \( \Omega_\delta^0 \) and \( \Omega_0^0 \). To this end, we introduce a transformation of the independent variables:

\[
x_1 = y_1, \quad x_2 = y_2 - \delta \theta(y) \psi(y_1), \quad x \in \Omega_0^0, \quad y \in \Omega_\delta^0 \tag{2.13}
\]

with \( \theta \in C_0^\infty(\Omega) \), \( \theta = 1 \) in a neighborhood of \( \Gamma_\delta^c \). Remind that \( \psi \in \Psi \subseteq H^2(a, b) \), hence we can extend the function \( \psi \) beyond \( (a, b) \) to the interval \( (A, B) \), where \( A < a \) and \( b < B \). Without any restrictions the function \( \psi \) is assumed to belong to the space \( H^2_0(A, B) \), i.e., it has zero values \( \psi(A), \psi_A(A), \psi(B), \psi_A(B) \), where the index \( y_1 \) means the derivative \( \psi' \). Consequently, the extended function \( \psi \) can be extended once again outside the interval \( (A, B) \) by zero to have a correct definition of the map (2.13). It is important to note at this point that the following estimate holds:

\[
\|\psi\|_{H^2_0(A, B)} \leq c\|\psi\|_{H^2(a, b)} \quad \forall \psi \in \Psi \tag{2.14}
\]

with some constant \( c > 0 \). Of course in the above considerations the set \( (A, B) \times \{0\} \) is assumed to belong to \( \Omega \).

Let \( u_\delta(x) = u^\delta(y) \), \( y \in \Omega_\delta^0, \ x \in \Omega_0^0 \). We prove the following assertion on the strong convergence of solutions \( u_\delta \).

**Lemma 2.1.** Let \( u \) be a solution of the problem (2.12). Then as \( \delta \to 0 \), \( u_\delta(x) \to u(x) \) strongly in \( H^{1,0}(\Omega_0^0) \).
Proof. Consider the Jacobian $g_δ(y) = |∂x(y)/∂y|$ of the transformation (2.13). It is clear that $g_δ(y) = 1 - δψθy > 0$ for small $δ$. Denote $h_δ(y) = g_δ^{-1}(y)$. We change the domain of integration $Ω_δ^0$ by $Ω_0^0$ in (2.11) in accordance with (2.13). This provides the relation

$$u_δ ∈ K_0: \int_{Ω_0^0} ∇u_δ(∇u_δ - ∇u_δh_δ) + δ \int_{Ω_0^0} G(u_δ^2, u_δ, u_δx, δ, δ(ψθ) )h_δ \geq \int_{Ω_0^0} \tilde{f}(u_δ - u_δ)h_δ \quad ∀u_δ ∈ K_0.$$  (2.15)

Here $\tilde{f}(x) = f(y(x))$, and we use the following formulae for the first derivatives,

$$u_δ^y_1 = u_δx_1 - δu_δx_2(θψ)_y, \quad u_δ^y_2 = u_δx_2(1 - δθy).$$

with the above notations, $u_δ(y) = u_δ(x), y ∈ Ω_δ^0, x ∈ Ω_0^0$. The function $G$ linearly depends on $u_δ^2, u_δ, u_δx$ and, in particular, as $δ → 0$,

$$δ \int_{Ω_0^0} G(u_δ^2, u_δ, u_δx, δ, δ(ψθ) )h_δ → 0.$$  (2.16)

provided that $u_δ, u_δ$ are bounded in $H^{1,0}(Ω_0^0)$ uniformly in $δ$. Let us prove the boundedness of $u_δ$ in the space $H^{1,0}(Ω_0^0)$. We take $\tilde{u}_δ = 0$ in (2.15). This yields the inequality

$$\int_{Ω_0^0} |∇u_δ|^2h_δ ≤ δ \int_{Ω_0^0} G(u_δ^2, 0, 0, δ(ψθ) )h_δ + \int_{Ω_0^0} \tilde{f}u_δh_δ.$$  (2.17)

Note that for small $δ$ the inequality $1/2 < h_δ < 3/2$ holds. Consequently from (2.17) it follows that uniformly in $δ$

$$\|u_δ\|_{H^{1,0}(Ω_0^0)} ≤ c.$$  (2.18)

Now we substitute $\tilde{u} = u_δ$ in (2.12) and $\tilde{u}_δ = u$ in (2.15). Summing the relations obtained in such a way the following inequality is derived:

$$\int_{Ω_0^0} (∇u - (∇u - ∇u_δh_δ) - δ \int_{Ω_0^0} G(u_δ^2, u_δ, u_δx, δ, δ(ψθ) )h_δ \leq \int_{Ω_0^0} (u - u_δ)(f - \tilde{f}h_δ).$$  (2.19)
Note that
\[
\|f - \tilde{f}h_\delta\|_{L^2(\Omega^0_\delta)} \leq c\delta \quad (2.20)
\] with a constant \(c\) being uniform in \(\delta\). Since
\[
\int_{\Omega^0_\delta} (\nabla u - \nabla u_\delta)(\nabla u - \nabla u_\delta) = \int_{\Omega^0_\delta} |\nabla u - \nabla u_\delta|^2 h_\delta - \delta \int_{\Omega^0_\delta} \psi\theta_\delta h_\delta \nabla u(\nabla u - \nabla u_\delta)
\] and \(h_\delta > 1/2\) it follows from (2.18), (2.19), (2.20) that
\[
\|u_\delta - u\|_{H^{1,0}(\Omega^0_\delta)} \to 0, \quad \delta \to 0.
\] In fact, we obtain an existence of a constant \(c\) such that
\[
\|u_\delta - u\|_{H^{1,0}(\Omega^0_\delta)} \leq c\delta, \quad \delta \to 0.
\] Lemma 2.1 is proved.

For any fixed \(\delta\), i.e., for any fixed crack shape \(y_2 = \delta\psi(y_1), \psi \in \Psi\), in accordance with (1.9), we can find the derivative of the energy functional with respect to the perturbation parameter \(\varepsilon\). Thus, the following formula for the derivative of the energy functional with respect to \(\varepsilon\) as \(\varepsilon = 0\) can be obtained:
\[
E'(\delta\psi) = \int_{\Omega^0_\delta} \left\{ \frac{1}{2} |\nabla u_\delta|^2 \div V - u_\delta^i u_\delta^j V^p \right\} - \int_{\Omega^0_\delta} u_\delta^i \div (f V), \quad (2.21)
\] We can write the formula for the derivative of the energy functional for the problem (2.12) which gives:
\[
E'(0) = \int_{\Omega^0_n} \left\{ \frac{1}{2} |\nabla u|^2 \div V - u^i u^j V^p \right\} - \int_{\Omega^0_n} u \div (f V), \quad (2.22)
\] where \(u\) is the solution of (2.12). Now change the integration domain \(\Omega^0_\delta\) by \(\Omega^0_n\) in (2.21) in accordance with (2.13). Note that the inequality (2.12) follows from (2.15) as \(\delta \to 0\). Consequently, by Lemma 2.1, we have a strong convergence of solutions \(u_\delta(y) = u_\delta(x)\) in the space \(H^{1,0}(\Omega^0_\delta)\) and we derive:
\[
E'(\delta\psi) \to E'(0), \quad \delta \to 0.
\] So we have obtained the continuity of the derivative of the energy functional with respect to the crack shape. In particular, the convergence (2.23) shows the continuity of the derivative with respect to the domain shape.
In the sequel to underline the dependence of geometrical domains and other sets on $\psi \in \Psi$ we shall use the following notations. Let $I_0^\psi$ be the graph of the function $y_2 = \psi(y_1)$, $y_1 \in (a, b)$. $I_\varepsilon^\psi = \varphi_\varepsilon(I_0^\psi)$, $\Omega_0^\psi = \Omega \setminus \overline{I_0^\psi}$, $\Omega_\varepsilon^\psi = \varphi_\varepsilon(\Omega_0^\psi)$. Also we introduce the set of admissible displacements for the unperturbed and perturbed problems, respectively,

$$K_0^\psi = \{v \in H^{1,0}(\Omega_0^\psi) | [v] \geq 0 \text{ on } I_0^\psi\},$$
$$K_\varepsilon^\psi = \{v \in H^{1,0}(\Omega_\varepsilon^\psi) | [v] \geq 0 \text{ on } I_\varepsilon^\psi\}.$$

Now we are in a position to prove the following result concerning an existence of optimal domain shape.

**Theorem 2.1.** There exists a solution of the optimal control problem (2.10).

**Proof.** Let $\psi^m \in \Psi$ be a minimizing sequence in the problem (2.10). Since the set $\Psi$ is bounded in $H^2(a, b)$ we can assume that as $m \to \infty$,

$$\psi^m \to \psi \quad \text{weakly in } H^2(a, b), \quad \psi^m_{y_1} \to \psi_{y_1} \quad \text{in } C[a, b]. \tag{2.24}$$

For any fixed $m \in N$ we can find the solution $u^m$ of the problem

$$u^m \in K_0^m: \int_{\Omega_0^m} \nabla u^m (\nabla \overline{u} - \nabla u^m) \geq \int_{\Omega_0^m} f(\overline{u} - u^m) \quad \forall \overline{u} \in K_0^m. \tag{2.25}$$

Here the domains $\Omega_0^m$ correspond to the graphs of functions $y_2 = \psi^m(y_1)$, respectively, i.e., $\Omega_0^m = \Omega \setminus \overline{I_0^m}$. Similarly, the sets $K_0^m$ and $I_0^m$ fit to the same graphs of the functions $y_2 = \psi^m(y_1)$.

Let us change the variables:

$$x_1 = y_1, \quad x_2 = y_2 + \theta(y)(\psi(y_1) - \psi^m(y_1)), \tag{2.26}$$

where $y \in \Omega_0^m$, $x \in \Omega_\varepsilon^\psi$, and the function $\theta$ is from $C^\infty_c(\Omega)$, $\theta = 1$ in a neighborhood of the graph of the function $y_2 = \psi(y_1)$, $y_1 \in (A, B)$. All functions $\psi \in \Psi$ are extended beyond $(a, b)$ to the interval $(A, B)$, $A < a, b < B$, in such a way that $\psi(y_1) = \psi_{y_1}(y_1) = 0$ for $y_1 = A, B$. Also we assume that the extended functions $\psi$ are equal to zero outside $(A, B)$.

Hence the definition (2.26) is correct.

Now we find the derivative of the energy functional with respect to $\varepsilon$ for a given $m \in N$.

This gives:

$$E'(\psi^m) = \int_{\Omega_0^m} \left\{ \frac{1}{2} |\nabla u^m|^2 \text{div } V - u^m_{n,p} V_{\varepsilon,p} \right\} - \int_{\Omega_0^m} u^m (\text{div } f V), \tag{2.27}$$
where $u^m$ solves the problem (2.25). Analogously, for the limit function $\psi$ we can get the formula for the derivative of the energy functional:

$$E'(\psi) = \int_{\Omega^\psi_0} \left\{ \frac{1}{2} |\nabla u|^2 \text{div} V - u \cdot u_p V_p \right\} - \int_{\Omega^\psi_0} u \text{div}(fV),$$

where $u$ is the solution of the problem

$$u \in K_0^\psi: \int_{\Omega^\psi_0} \nabla u(\nabla \bar{u} - \nabla u) \geq \int_{\Omega^\psi_0} f(\bar{u} - u) \quad \forall \bar{u} \in K_0^\psi. \quad (2.29)$$

Finding the derivative (2.28) means that we consider the perturbation of the problem (2.29) with respect to the parameter $\varepsilon$ and solve the problem:

$$u^\varepsilon \in K_0^\psi: \int_{\Omega^\psi_0} \nabla u^\varepsilon(\nabla \bar{u} - \nabla u^\varepsilon) \geq \int_{\Omega^\psi_0} f(\bar{u} - u^\varepsilon) \quad \forall \bar{u} \in K_0^\psi. \quad (2.30)$$

Introduce the energy functional for this problem:

$$E(\Omega^\psi_\varepsilon) = \frac{1}{2} \int_{\Omega^\psi_\varepsilon} |\nabla u^\varepsilon|^2 - \int_{\Omega^\psi_\varepsilon} f u^\varepsilon.$$

Then the right-hand side of (2.28) is equal to $dE(\Omega^\psi_\varepsilon)/d\varepsilon|_{\varepsilon=0}$.

Similar to Lemma 2.1, it can be proved that

$$u_m \rightarrow u \quad \text{strongly in } H^{1,0}(\Omega^\psi_0), \quad (2.30)$$

where $u_m(x) = u^m(y), y \in \Omega^m_0, x \in \Omega^\psi_0$. We can change the domain of integration $\Omega^m_0$ by $\Omega^\psi_0$ in (2.27) in accordance with (2.26). Analogously to (2.23) the convergences (2.24), (2.30) allow us to pass to the limit as $m \rightarrow \infty$ in the relation obtained. This provides the convergence

$$E'(\psi^m) \rightarrow E'(\psi).$$

Since $\psi \in \Psi, u = u(\psi)$, the limit function $\psi$ solves the problem (2.10). Theorem 2.1 is proved. \qed

Note that instead of (1.2) we can consider other boundary conditions at the external boundary $\Gamma_0$. In particular, let $\Gamma_0 = \Gamma_0^1 \cup \Gamma_0^2, \Gamma_0^1 \cap \Gamma_0^2 = \emptyset, \text{meas } \Gamma_0^1 > 0$. Denote by $n$ the unit normal vector to $\Gamma_0$. In this case we can impose the following conditions:

$$u = 0 \quad \text{on } \Gamma_0^1, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_0^2.$$
Moreover, we can apply the arguments presented above to the linear crack model and find a domain shape providing minimal derivative of the energy functional. In the linear crack model instead of (1.3) we should consider the following boundary conditions:

$$\frac{\partial u}{\partial n} = 0$$ on \( \Gamma_{G_c}^{\pm} \).

The result of the paper provides an existence of the optimal domain shape for the proposed criterion. This means that having chosen any fixed function \( \psi \in \Psi \) we increase the derivative of the energy functional as compared to the optimal case. Let \( \psi_0 \in \Psi \) be a solution of the problem (2.10) and

$$E'(\psi_0) = \frac{dE(\Omega_{\psi_0}^{\delta})}{d\epsilon} \bigg|_{\epsilon=0}, \quad E'(\psi) = \frac{dE(\Omega_{\psi}^{\delta})}{d\epsilon} \bigg|_{\epsilon=0}.$$  

The above arguments show that \( E'(\psi_0) \leq E'(\psi) \) and consequently the crack shape \( y_2 = \psi(y_1) \) would be more favorable as compared to the shape \( y_2 = \psi_0(y_1) \) provided that we use the generalized Griffith criterion.

### 3. Examples and related problems

To illustrate Theorem 2.1 we provide an example. Consider the case of linear crack shapes of the following form:

$$\Psi = \{ \delta \psi \mid \psi(y_1) = d, \ y_1 \in (a, b), \ |\delta| \leq \delta_0 \},$$

where \( d, \delta_0 \in \mathbb{R} \) are given numbers.

Let \( \Omega_0^\delta \) be a domain corresponding to the graph \( \Gamma_{G_c}^\delta \), i.e., to the graph of the function \( y_2 = \delta \psi(y_1) \). Assume that \( \varphi_\varepsilon \) describes a crack length change for small \( \varepsilon \). This means

$$\varphi_\varepsilon(\Omega_0^\delta) = \Omega_\varepsilon^\delta$$

with

$$\Omega_\varepsilon^\delta = \Omega_0^\delta \setminus \varphi_\varepsilon(\Gamma_{G_c}^\delta).$$

and \( \varphi_\varepsilon(\Gamma_{G}^\delta) \) is the graph of the function \( y_2 = \delta \psi(y_1) \), \( \psi(y_1) = d \), \( y_1 \in (a, b + \varepsilon) \). Consequently, we consider, in fact, a crack length change through the perturbation of the crack tip \( (b, d\delta) \). Hence in this case Theorem 2.1 provides an existence of an optimal crack shape \( y_2 = \delta \psi(y_1), \ y_1 \in (a, b) \), with some fixed value \( \delta_0 \in [-\delta_0, \delta_0] \). Moreover, we can write down analytically the map \( \varphi_\varepsilon \) in this case. Let \( \xi \in C_0^\infty(\Omega), \xi = 1 \) in a neighborhood of the set \( \{b\} \times (\delta_0|d|, \delta_0|d|) \). Consider a perturbation of the domain \( \Omega_0^\delta \) through the transformation of the independent variables:

$$x_1 = y_1 + \varepsilon \xi(y_1, y_2), \quad x_2 = y_2; \quad (y_1, y_2) \in \Omega_0^\delta, \quad (x_1, x_2) \in \Omega_\varepsilon^\delta.$$
Hence \( \varphi_\varepsilon(y) = (y_1 + \varepsilon \xi(y_1, y_2), y_2) \) and we obtain the formula for the vector-field \( V(y) \),

\[
V(y) = (\xi(y_1, y_2), 0).
\]

By (1.9), the derivative of the energy functional is equal to:

\[
E'(\delta \psi) = \frac{1}{2} \int_{\Omega_0} (\xi, (u_2^2 - u_1^2) - 2\xi_2 u_1 u_2, f, 1u)
\]

where \( u = u^\delta \) is the solution of the problem like (2.11) with the above crack \( \Gamma_c^\delta \). We should note that this derivative does not depend on the chosen function \( \xi \) with the prescribed properties. According to Theorem 2.1 we have:

\[
E'(\delta_s \psi) = \min_{|\delta| \leq \delta_0} E'(\delta \psi). 
\]

We should remark that the perturbation \( x = \varphi_\varepsilon(y) \) is fixed in our considerations. Analyzing the optimal control problem (2.10) we select the best crack shape \( y_2 = \psi(y_1) \), \( y_1 \in (a, b) \). For each \( \psi \in \Psi \) the vector-field \( V(y) \) is the same since this field is determined by the perturbation \( x = \varphi_\varepsilon(y) \). We can consider the case when the perturbation \( x = \varphi_\varepsilon(y) \) depends on the function \( \psi \in \Psi \).

Indeed, assume that for each \( \psi \in \Psi \) we consider a linear extension outside the right tip \( (b, \psi(b)) \) of the graph of the function \( y_2 = \psi(y_1), y_1 \in (a, b) \). Also let \( \xi \in C_0^\infty(\Omega) \), \( \xi = 1 \) in a neighborhood of the point \( (b, 0) \).

Let \( \theta \in C_0^\infty(\Omega), \theta = 1 \) in a neighborhood of graphs of the functions \( y_2 = \psi(y_1), y_2 = 0, y_1 \in (a, b), \psi \in \Psi \). Also let \( \xi \in C_0^\infty(\Omega), \xi = 1 \) in a neighborhood of the point \( (b, 0) \).

Denote \( p(y) = y_2 - \psi(y_1)\theta(y) \) and introduce the perturbation \( x = \varphi_\varepsilon(y) \) by the formula:

\[
\begin{align*}
x_1 &= y_1 + \varepsilon \xi(y_1, p(y)), \\
x_2 &= p(y) + \psi(y_1) + \varepsilon \xi(y_1, p(y))\theta(y_1 + \varepsilon \xi(y_1, p(y))) p(y) \end{align*}
\]

(3.31)

In so doing we extend the functions \( \psi \) outside \( (a, b + \varepsilon) \) to the interval \( (A, B) \) similar to that of Theorem 2.1.

From (3.31) it follows that the vector-field \( V(y) = d\varphi_\varepsilon(y)/d\varepsilon|_{\varepsilon=0} \) depends on \( \psi \), \( \psi \in \Psi \), since

\[
\begin{align*}
V^1(y) &= \xi(y_1, p(y)), \\
V^2(y) &= \psi(y_1)\xi(y_1, p(y))\theta(y_1, p(y)) + \psi(y_1)\theta_1(y_1, p(y))\xi(y_1, p(y)).
\end{align*}
\]

(3.32)

Consider the optimal control problem:

\[
\min_{\psi \in \Psi} E'(\psi),
\]

(3.33)
where $E'(\psi)$ is given by the formula (1.9) with the vector-field $V(y)$ from (3.32).

There exists a solution of the problem (3.33). The scheme of the proof follows that of Theorem 2.1.

We consider an example where the vector-field $V(y)$ depends on $\psi$. Let

$$\Psi = \{ \delta \psi \mid \psi(y_1) = -\frac{b}{a} y_1 + b, \ y_1 \in (a, b), \ |\delta| \leq \delta_0 \},$$

where $\delta_0 \in \mathbb{R}$ is a given number. Denote by $\Omega^\delta_0$ a domain corresponding to the graph $\Gamma^c_\delta$, i.e., to the graph of the function $y_2 = \delta \psi(y_1)$. Let $\varphi_{\epsilon}$ describe a crack length change for small $\epsilon$. We have:

$$\varphi_{\epsilon}(\Omega^\delta_0) = \Omega^\delta_\epsilon, \ \Omega^\delta_\epsilon = \Omega^\delta_0 \setminus \varphi_{\epsilon}(\Gamma^c_\delta)$$

and $\varphi_{\epsilon}(\Gamma^c_\delta)$ is the graph of the function $y_2 = \delta \psi(y_1)$, $\psi(y_1) = -\frac{b}{a} y_1 + b$, $y_1 \in (a, b + \epsilon)$. Thus, we consider a crack length change through the perturbation of the crack tip $(b, -\frac{b^2}{a} \delta + b\delta)$. In this case the vector-field $V(y)$ depends on the function $\delta \psi$.

We can consider other cost functionals as compared to derivatives of energy functionals $E'(\psi)$. In particular, there are a number of functionals which do not require a consideration of the perturbation of the domain $\Omega_0$ through the map $\varphi_{\epsilon}(y)$.

Consider two examples.

Let $u \in K_0^\psi$ be a solution of the problem (2.29) for a given $\psi \in \Psi$. Introduce the cost functional which characterizes an opening of the crack:

$$\Pi_1(\psi) = \int_{\Gamma^\psi_0} |u|$$

and consider the optimal control problem for finding a domain shape

$$\min_{\psi \in \Psi} \Pi_1(\psi). \quad (3.34)$$

It can be proved that the problem (3.34) has a solution. Indeed, let $\psi^m \in \Psi$ be a minimizing sequence in the problem (3.34). Since the set $\Psi$ is bounded in $H^2(a, b)$ we can assume that as $m \to \infty$

$$\psi^m \to \psi \quad \text{weakly in } H^2(a, b), \quad \psi^m_{y_1} \to \psi_{y_1} \quad \text{in } C[a, b]. \quad (3.35)$$

Then, like in the proof of Theorem 2.1, we have the strong convergence for solutions $u^m$ corresponding to $\psi^m$:

$$u^m \to u \quad \text{strongly in } H^{1,0}(\Omega^\psi_0) \quad (3.36)$$

with the usual notations, $u^m(x) = u^m(y)$, $y \in \Omega^m_0, x \in \Omega^\psi_0$,

$$x_1 = y_1, \quad x_2 = y_2 + \theta(y)\left(\psi(y_1) - \psi^m(y_1)\right), \quad (3.37)$$
and the function \( \theta \) belongs to \( C^\infty_0(\Omega) \), \( \theta = 1 \) in a neighborhood of the graph of the function \( y_2 = \psi(y_1), y_1 \in (a, b) \). We extend the functions \( \psi, \psi^m \) outside \( (a, b) \) similar to that in the proof of Theorem 2.1. Hence, by (3.35), (3.36),

\[
\Pi_1(\psi^m) = \int_{\Gamma_0^\psi} |u^m| \rightarrow \int_{\Gamma_0^\psi} |u|.
\]

Since \( u = u(\psi) \) we obtain:

\[
\Pi_1(\psi) = \int_{\Gamma_0^\psi} |u|
\]

and \( \psi \in \Psi \) solves the problem (3.34).

Now consider the other cost functional. Again, let \( u \in K^\psi_0 \) solve the problem (2.29) with a given \( \psi \in \Psi \). We introduce the functional:

\[
\Pi_2(\psi) = \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma_0^\psi)}.
\]

Remind that \( \Gamma_0^\psi \) is the external boundary of the domain \( \Omega \), and \( n \) is the unit normal vector to \( \Gamma_0^\psi \). Since \( \Gamma_0^\psi \) is the smooth boundary there exists a strip \( \Omega_h \) near \( \Gamma_0^\psi \) with a given positive thickness \( h \) such that the solution \( u \) of the problem (2.29) has the property

\[
u \in H^2(\Omega_h).
\]

Hence the functional (3.38) is defined correctly. Consider the optimal control problem:

\[
\min_{\psi \in \Psi} \Pi_2(\psi).
\]

There exists a solution of the problem (3.39). To prove this statement we choose a minimizing sequence \( \psi^m \in \Psi \) in the problem (3.39). It can be assumed that, as \( m \rightarrow \infty \),

\[
\psi^m \rightarrow \psi \quad \text{weakly in} \quad H^2(a, b), \quad \psi^m_{y_1} \rightarrow \psi_{y_1} \quad \text{in} \quad C[a, b].
\]

The solutions \( u^m \) are defined from the variational inequality:

\[
u^m \in K_0^m: \ \ \int_{\Omega_0^m} \nabla u^m (\nabla \bar{u} - \nabla u^m) \geq \int_{\Omega_0^m} f(\bar{u} - u^m) \quad \forall \bar{u} \in K_0^m.
\]

We have:

\[
-\Delta u^m = f \quad \text{in} \ \Omega_b, \quad u^m = 0 \quad \text{on} \ \Gamma_0.
\]
Again, the strong convergence of \( u_m(y) = u_m(x), \ y \in \Omega^m_0, \ x \in \Omega^\psi_0 \), takes place, i.e.,

\[
u_m \to u \quad \text{strongly in } H^{1,0}(\Omega^\psi_0).
\]

Relations (3.40) imply

\[
-\Delta u_m = f \quad \text{in } \Omega_h, \quad u_m = 0 \quad \text{on } \Gamma_0
\]

and, by the regularity results for elliptic equations,

\[
\|u_m\|_{H^2(\Omega_h/2)} \leq c \|f\|_{L^2(\Omega)} \quad (3.41)
\]

with a constant \( c > 0 \) being uniform with respect to \( m \). We assume that the strip \( \Omega_h \) is small enough to have an empty intersection with both \( \text{supp} \theta \), where the function \( \theta \) is taken from (3.37), and with graphs of the functions \( y_2 = \psi(y_1), \ y_1 \in (a, b), \ \psi \in \Psi \). Hence, by (3.41),

\[
u_m \to u \quad \text{weakly in } H^2(\Omega_h/2),
\]

\[
\Pi_2(\psi^m) = \left\| \frac{\partial u_m}{\partial n} \right\|_{L^2(\Gamma_0)} \to \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma_0)}.
\]

To conclude the proof of the statement we should take into account the equality \( u = u(\psi) \) which implies

\[
\Pi_2(\psi) = \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma_0)}.
\]

Hence \( \psi \) is the solution of the problem (3.39).

**Acknowledgements**

This work was done during the visit of A. Khludnev to Federal University of Rio de Janeiro in June–July 2002. The authors acknowledge the financial support provided by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico, Brasil), FAPERJ (Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro) and FUJB (Fundação Universitária José Bonifácio, Rio de Janeiro). The first author (A. Kh.) thanks also the Russian Fund for Basic Research (00-01-00842) and the Russian Ministry of Education (2000.4.19).
References